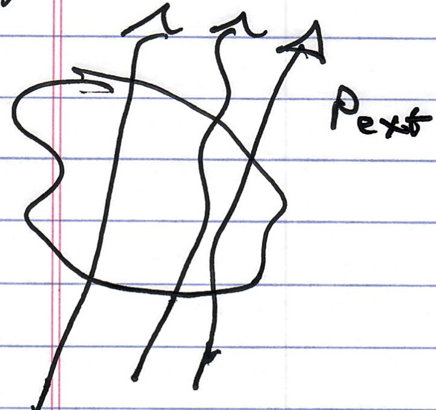
Conservation Laws, cont'd.

Recall: Scalar Virial Theorem, with self-gravitation,

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_{kin} + 2E_p - |E_g| + E_B$$

- obviously cannot self-confine without self-gravity
- from tr(Tensor Virial Thm)
- closed system.

Now, consider plasma cloud / blob



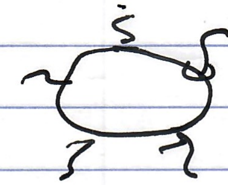
- mass M, radius R
- threaded by $\underline{B}$
- external pressure P_0
- no bulk motion
- frozen flux

Does cloud collapse under gravity → star

⇒ inertial confinement fusion that works

Now, easy to show for $\ddot{I} = 0$, $\tilde{v} \to 0$ $(E_k \to 0)$

P_0



$$\left[2E_p - |E_g| + E_B\right]$$

$$= \int dA\, \underbrace{P_{ext}}_{\text{external pressure}}\, \underline{x}\cdot\hat{n} - \int dA\, \underline{x} \cdot \underbrace{\underline{\underline{T}}_B}_{\text{magnetic stress on surface, by threading field}} \cdot \hat{n}$$

show This !

Now, at estimate level:

$$M = \int \rho\, dV \sim \rho R^3 \rightarrow M$$

$$E_p \sim c_s^2 M$$

$$|E_g| \sim \underbrace{\alpha}_{\text{form factor}} \frac{GM^2}{R}$$

For frozen flux Chemiosphere)

$$\Phi \sim 2\pi R^2 B$$

so $$E_B + \int dA \, \underline{x} \cdot \underline{\underline{T}}_B \cdot \vec{n} \sim \frac{\beta \Phi^2}{R}$$

form factor

so have, after eliminating extraneous factors:

$$R^3 P_{ext} \sim \left(\frac{\beta \Phi^2}{R} - \alpha \frac{GM^2}{R} + \frac{3}{2} c_s^2 M \right)$$

~ scalar virial theorem for cloud.

$\Rightarrow$

note similar scaling

$$P_{ext} \sim \left(\frac{\beta \Phi^2}{R^{\cancel{3}4}} - \alpha \frac{GM^2}{R^{\cancel{3}4}} + \frac{3}{2} \frac{M c_s^2}{R^3} \right)$$

N.B.:

– Φ, $G \to 0$ $\qquad P_{ext} = P_{int}$

$\qquad\qquad\qquad\qquad \doteq \rho c_s^2$

– if $\Phi = 0$

then

$$P_{ext} \sim \frac{-\alpha\, G M^2}{R^4} + \frac{3}{2}\frac{M c_s^2}{R^3}$$

$$\frac{dP}{dR} = 0 \quad \rightarrow \quad \text{equilibrium}$$

$$\frac{4\alpha\, G M^2}{R^5} \sim \frac{9}{2}\frac{M c_s^2}{R^4}$$

$$R_{max} \cong G M \alpha / c_s^2$$

↓

Radius at which P maximal

Maximal P: $R_{max} \cong G M \alpha / c_s^2$

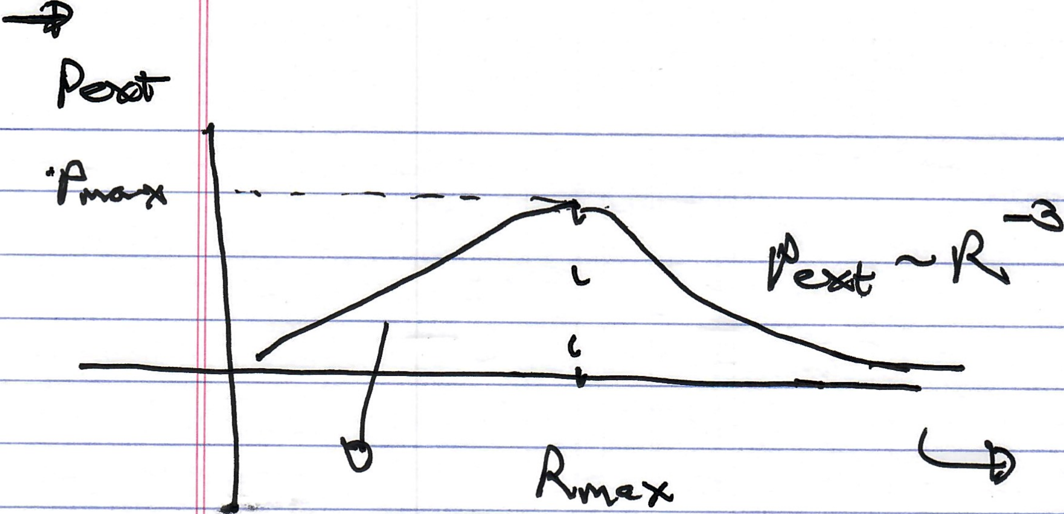
- re-write:

$$R_m \cong G R_m^3 \rho \alpha / c_s^2$$

$$R_m^2 = L_{Jeans}^2 \sim (c_s^2 / G\rho)$$

↓

Jeans Length → marginal scale.



self-confinement
→ external pressure not needed

↳ pressure decays, P_{ext} needed drops.

→ $P > P_{max}$ → no eqbrm

→ $R < R_{max}$ → P_{ext} decreases to maintain equilibrium
→ instability to gravitational collapse

→ Now, $\Phi \neq 0$ ⇒ note that magnetic support scales similarly to gravitational attraction, with opposite sign.

$$\boxed{P_{ext} \sim \frac{\beta \Phi^2 - \alpha G M^2}{R^4} + \frac{3}{2}\frac{C_s^2 M}{R^3}}$$

so, key comparison is:

$$M \lessgtr M_\Phi = \sqrt{\beta/\alpha}\; \Phi / G^{1/2}$$

ⓐ $M < M_\Phi$ → magnetic support wins, magnetically subcritical mass for grav. collapse

→ repulsive effects win always

→ no amount external compression induces indefinite contraction if flux remains frozen in.

ⓑ $M > M_\Phi$ → magnetically supercritical mass for collapse

→ sufficient external pressure can induce collapse, even if flux frozen in.

NB: If kinetic energy contribution, NL Alfven effect / waves can support cloud

For perspective, recall:

- Chandrasekhar Mass

  $\rightarrow$ $M > M_{Chandra}$ $\rightarrow$ collapse

  $\rightarrow$ $M < M_{Chandra}$ $\rightarrow$ no collapse

Recall

$M_{Chandra}$ derived for degenerate Fermi gas equation of state $\rightarrow$ $\gamma = 4/3$ instead $\gamma = 5/3$

if flux freezing:

$$\Phi \sim \rho R^2 B \qquad M \sim \rho R^3$$

$$B \sim R^{-2} \qquad \rho \sim M/R^3$$

$$\sim \rho^{2/3}$$

$$\boxed{P_{mag} \sim B^2 \sim \rho^{4/3}}$$

magnetic pressure E-o-S similar to

degenerate Fermi gas.

→ Analogue to Chandrasekhar Mass plausible.

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

$\therefore$ $n_{\text{Fermion}} \sim N/R^3$

$\therefore$ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$p \sim \hbar/\Delta x \sim \hbar\, n^{1/3}$ (Heisenberg Uncertainty)

p → Fermion Momentum

⇒ Fermion energy (per Fermion): $E_F = pc$

$\sim \hbar c \dfrac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

Gravitational Energy (per Fermion): $E_{Grav} \sim -\dfrac{GM}{R} m_b$

m_b → Baryon mass

$M \sim N\, m_B$

Pressure → electron
Mass → Baryon

$\therefore$ $E = E_F + E_G$

$= \dfrac{\hbar c N^{1/3}}{R} - \dfrac{G N m_B^2}{R}$

note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,
$\therefore E_F \sim 1/R^2 \rightarrow$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse

$\therefore$ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore \quad M_{Chandra} = N_{max} m_B \sim 1.5 M_\odot$$