Next: $\boxed{\text{Conservation Laws}}$ and Virial Theorem

→ Conservation Laws in MHD

– here discuss: conservation $\begin{cases} \text{momentum} \\ \text{energy} \\ \text{angular momentum} \end{cases}$

and virial theorems.

→ Momentum → key: construct evolution of momentum density.

have:
$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi} + \underbrace{\rho\underline{g}}_{\text{body force}}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}(\rho\underline{v})}_{\text{momentum density}} + \nabla\cdot(\underbrace{\rho\underline{v}\,\underline{v}}_{\text{Reynolds stress tensor}}) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \nabla\cdot\frac{\underline{B}\,\underline{B}}{4\pi} + \rho\underline{g}$$

$$\underline{T}_R = \rho\underline{v}\,\underline{v}$$

Maxwell stress tensor
$$T_B = \frac{B^2}{8\pi}\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}.$$

thus re-write:

$$\frac{\partial}{\partial t}(\rho\underline{v}) = -\nabla\cdot\underline{\underline{T}} + \rho\underline{g}$$

where

$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho\,\underline{V}\,\underline{V}$$

$$T_{ij} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho V_i V_j$$

→ ala Gaussian surface

Then, if consider a 'blob' of $\left\{\begin{array}{l}\text{plasma}\\ \text{magnetofluid}\end{array}\right.$:

surface S — crucial

blob enclosed by arbitrary, non-dynamical surface.

momentum density

$$\underbrace{\frac{\partial}{\partial t}\underline{P}}_{\text{momentum}} = \int d^3x\, \frac{\partial}{\partial t}(\rho \underline{V})$$

$$= -\int d^3x\, \nabla\cdot\underline{\underline{T}} + \underbrace{\int d^3x\, \rho\,\underline{g}}_{\text{net body force}}$$

$$= -\int d\underline{S}\cdot\underline{\underline{T}} + \int d^3x\, \rho\,\underline{g}$$

$\boxed{\text{So, apart from volume integrated body force,}}$

$$\boxed{\frac{\partial P}{\partial t} = -\int d\underline{S}\cdot\underline{\underline{T}}}$$

{ change in momentum set by stress on surface of blob

$$\boxed{\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho\,\underline{V}\,\underline{V}}$$

Thus, can identify ways momentum is lost by the blob:

- $\underline{\underline{T}}_R \cdot d\underline{S} = +\rho \underline{V}\underline{V} \cdot d\underline{S}$ → flux of momentum density thru surface

- $\underline{\underline{T}}_{P_{tot}} \cdot dS = + \left(P + \frac{B^2}{8\pi}\right) \cdot d\underline{S}$ → pressure (total) force on surface, in $-d\underline{S}$ direction

- $\underline{\underline{T}}_{\substack{Mag\\ten}} \cdot dS = -\left(\frac{\underline{B}}{4\pi}\right) \underline{B} \cdot d\underline{S}$ → magnetic tension force in $+\underline{B}$ direction, piercing surface

$\sim (\underline{B} \cdot d\underline{S}) \frac{\underline{B}}{4\pi}$ → tension of $\underline{B}/4\pi$, per line

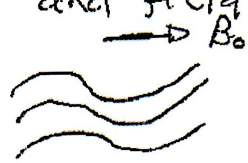
$(\underline{B} \cdot d\underline{S})$ → # of lines threads outward

i.e. 

→ Note that magnetic tension is independent of sign of $\underline{B}$ (as it should, tension is, strictly speaking, a dyad ⇕, not ↑)

↳ tension field $\sim \underline{B}\,\underline{B}$

→ can make obvious analogy between strings and field lines



#strings/area $= B$

$\sigma = \rho / B$ → mass per length of string

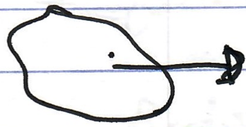
$T = B/4\pi$

Alfven wave

$v_{ph}^2 = T/\sigma$
$= B^2/4\pi\rho_0$
$= v_A^2$

Alfven

N.B.: $\underline{\underline{T}}_{magn} \cdot d\underline{s} = -\left(\frac{\underline{B}}{4\pi}\right)\left(\underline{B}\cdot d\underline{s}\right)$



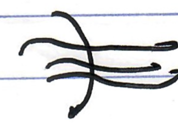
magnetic tension in $\underline{B}$ direction, piercing surface of blob

$$\underline{\underline{T}}_{magn} = \frac{\underline{B}\,\underline{B}}{4\pi}$$

$\Updownarrow$ dyad.

Note:

$$\underline{\underline{T}}_{magn\,ten} \cdot d\underline{s} = -\frac{\underline{B}}{4\pi}\left(\underline{B}\cdot d\underline{s}\right)$$



↳ # lines piercing surface

↓ tension of $B/4\pi$ per line

Now, recall for waves on a string, can define

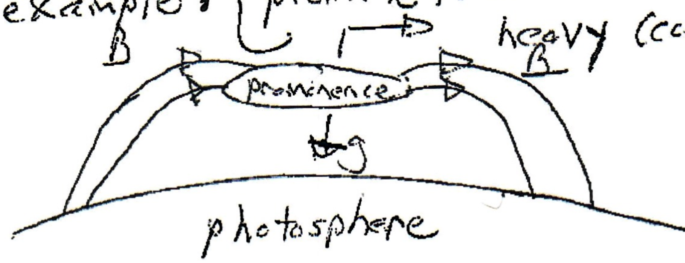
$$V_{ph}^2 = T/\mu$$

μ = mass per length

$T \Rightarrow B/4\pi$

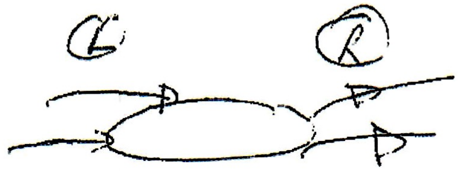
Observation.

→ example: {Solar prominence (see cover of Kulsrud) → heavy (cool) → requires support against gravity



{prominences often associated with radiative condensation

i.e.



L → #lines/area = $\underline{B}\cdot d\underline{S} < 0$ (inward)

force/Line is toward

∴ $\underline{F}_{L}$ → toward upper left

R → #lines/area = $\underline{B}\cdot d\underline{S} > 0$

f/Line is toward upper right

F_{R} → toward upper right

thus → prominence supported by magnetic tension (cf. hammock — string)

→ squashing $\underline{B}$ → support by magnetic pressure, too ....

→ The Skeptic: "what of EM Momentum ¿"

$$\underline{P}_{EM} \equiv \underline{E} \times \underline{B} / 4\pi c$$

$$E \sim \frac{VB}{c} \Rightarrow P_{EM} \sim (\rho V)\, B^2/4\pi\rho c^2$$

$$\sim \rho V \left(V_A^2/c^2\right) \ll 1 \quad \checkmark$$

n.b. obviously important in relativistic and EMHD, for $V_A \ll c$..

→ Angular Momentum → recall Kulsrud ---- "virtual surface"

→ $\boxed{\text{Energy}}$

Now energy: kinetic, thermal, magnetic, gravity

$$\boxed{E = E_V + E_p + E_B + E_g}$$

$$E = \int_V d^3x \left[\frac{1}{2}\rho V^2 + \frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \frac{\rho \phi}{2} \right]$$

self-gravitating blob

where $\begin{cases} \underline{g} = -\nabla\phi \\ \nabla^2\phi = 4\pi G\rho \end{cases}$ i.e. $\underline{g}$ evolves self-consistently (not "constant")

N.B. Problem: Jeans Instability

→ Calculate the growth rate of density perturbations in an un-magnetized, self-gravitating fluid

→ repeat in 1D, using Vlasov equation

→ Where does E_p come from?

Consider work to compress plasma/fluid, i.e.

$$dW = -p\,dV$$

$$\Delta E = -\int_0^{\rho_0} P(\rho)\, d(1/\rho) = \int_0^{\rho_0} (\rho/\rho_0)^{\gamma} P_0 \frac{d\rho}{\rho^2}$$

$$= \frac{P_0}{\rho_0(\gamma-1)} \Rightarrow \underset{\substack{\downarrow \\ \text{energy density}}}{\mathcal{E}} = \rho_0 \Delta E = \frac{P_0}{(\gamma-1)}$$

→ for energy balance, crank it out, using MHD equations ....

$$\frac{dE}{dt} = \underset{①}{\frac{d E_V}{dt}} + \underset{②}{\frac{d}{dt} E_p} + \underset{③}{\frac{d}{dt} E_B} + \underset{④}{\frac{d}{dt} E_g}$$

① $\frac{d}{dt} E_V = \int d^3x \, \underline{\frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right)}$ use $\underline{V}$ EOM

$$= \int d^3x \left[V^2 \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underline{V}}{\partial t} \cdot \underline{V} \right] d^3x$$

↳ ibp leaves S.T. and cancels 2nd.

$$= \int d^3x \Big[-\frac{\underline{V}^2}{2} \nabla\cdot(\rho \underline{V}) - \underline{V}\cdot \rho(\underline{V}\cdot\nabla\underline{V}) - \underline{V}\cdot\nabla P + \underline{V}\cdot(\underline{J}\times\underline{B}) - \rho\underline{V}\cdot\nabla\phi \Big]$$

i.e. $\int -\frac{V^2}{2}\nabla\cdot(\rho \underline{V}) = -\frac{V^2}{2}\rho \underline{V}\Big| + \int (\underline{V}\cdot\nabla\underline{V})\cdot\rho\underline{V}$

cancels 2nd term in $\frac{dE_V}{dt}$

② $\underline{\frac{d}{dt}E_p} = \int \frac{d^3x}{\gamma-1}\frac{\partial p}{\partial t}$ **EOS**

$\frac{d}{dt}(P/\rho^\gamma) = 0$

Now $\underline{\text{eqn. state}} \Rightarrow \frac{1}{p}\frac{dp}{dt} + \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$

$\left[\frac{1}{(P/\rho^\gamma)}\frac{d}{dt}(P/\rho^\gamma) = 0\right]$

and $\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\underline{V}$

$$\Rightarrow \frac{\partial p}{\partial t} = -\underline{V}\cdot\nabla p - \gamma p\nabla\cdot\underline{V}$$

So $\frac{d}{dt}E_p = \frac{-1}{(\gamma-1)}\int d^3x\,(\underline{V}\cdot\nabla p + \gamma p\nabla\cdot\underline{V})$

$$= -\int d^3x\left[\frac{\gamma}{\gamma-1}\nabla\cdot(p\underline{V}) - \underline{V}\cdot\nabla p\right]$$

yields a surface term

cancels $\underline{V}\cdot\nabla p$ term in $\frac{dE_n}{dt}$

expect similar relation between $\underline{J}\times\underline{B}$ and $\frac{\partial}{\partial t}B^2$ .... so...

③ $$\frac{d}{dt} E_B = \frac{1}{4\pi} \int d^3x \; \underline{B} \cdot \frac{\partial \underline{B}}{\partial t}$$

induct. →

$$= \frac{1}{4\pi} \int d^3x \; \underline{B} \cdot (\underline{\nabla} \times \underline{v} \times \underline{B})$$ by induction eqn.

ⓐ ⓑ

$$= -\int d^3x \left\{ \underline{\nabla} \cdot \left[\frac{\underline{B} \times (\underline{v} \times \underline{B})}{4\pi} \right] - \frac{(\underline{\nabla} \times \underline{B})}{4\pi} \cdot (\underline{v} \times \underline{B}) \right\}$$

surface term (→ Poynting)

$\underline{J} \cdot \underline{v} \times \underline{B}$

$$\text{ⓑ} = \int d^3x \; \underline{J} \cdot (\underline{v} \times \underline{B}) = -\int d^3x \; (\underline{J} \times \underline{B}) \cdot \underline{v}$$

cancels $\underline{v} \cdot \underline{J} \times \underline{B}$ term in dE_V/dt

which leaves:

④ $$\frac{dE_g}{dt} = \frac{1}{2} \int d^3x \left(\phi \frac{\partial \rho}{\partial t} + \rho \frac{\partial \phi}{\partial t} \right)$$

$$= \frac{1}{2} \int d^3x \; \phi \frac{\partial \rho}{\partial t} + \int d^3x \frac{\nabla^2 \phi}{8\pi G} \frac{\partial \phi}{\partial t}$$ ibp →

$$= \frac{1}{2} \int d^3x \; \phi \frac{\partial \rho}{\partial t} \, d^3x + \int \frac{\phi}{8\pi G} \nabla^2 \frac{\partial \phi}{\partial t} d^3x$$

so

$$\frac{dE_g}{dt} = \frac{1}{2}\int \phi \frac{\partial \rho}{\partial t} d^3x + \frac{1}{2}\int d^3x\, \phi \frac{\partial \rho}{\partial t}$$

$\frac{\partial \rho}{\partial t}$

$$= \int d^3x\, \phi \frac{\partial \rho}{\partial t} = -\int d^3x\, \phi\, \nabla\cdot(\rho \underline{V})$$

$$= +\int d^3x\, \rho \underline{V}\cdot \underline{\nabla}\phi \quad + -\int d\underline{S}\cdot \rho\phi \underline{V}$$

cancels $-\rho \underline{V}\cdot\nabla\phi$ in $\frac{dE_K}{dt}$

Note: $-\underline{V}\cdot\underline{\nabla} P$; $\underline{V}\cdot(\underline{J}\times\underline{B})$; $-\rho\underline{V}\cdot\underline{\nabla}\phi$; $\underline{V}\cdot\rho\underline{V}\cdot\underline{\nabla}\underline{V}$

terms <u>all</u> <u>cancel</u> in dE_V/dt !

Now adding up all <u>4</u> pieces $\Rightarrow$

$$\boxed{\frac{d}{dt}E = -\int d\underline{S}\cdot\Big[\underbrace{\rho\underline{V}\frac{V^2}{2}}_{\text{Conv. KE}} + \underbrace{\frac{\gamma}{\gamma-1}p\underline{V}}_{\gamma\text{ factor Conv. }p} - \underbrace{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}}_{\underline{E}\times\underline{B}} + \underbrace{\rho\underline{V}\phi}_{\underline{V}\phi}\Big]} \quad *$$

ST

i.e. not surprisingly, only survivors are surface terms.... $\Rightarrow$ in ideal MHD, only change in energy of blob involves boundary ...

i.e. have:

$$\frac{dE}{dt} = -\int d\underline{S} \cdot \left[\overset{①}{\rho \underline{V} \frac{V^2}{2}} + \overset{②}{\frac{\gamma}{\gamma-1} p \underline{V}} - \overset{③}{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}} + \overset{④}{\rho \underline{V} \phi} \right]$$

① → kinetic energy loss via simple kinetic energy flow thru surface.

② → $-\frac{\gamma \underline{V} \cdot d\underline{S}\, p}{\gamma-1}$ ⟷ outward flow of enthalpy

i.e. $-\frac{\gamma p \underline{V} \cdot d\underline{S}}{\gamma-1} = -\frac{p\, \underline{V} \cdot d\underline{S}}{\gamma-1} - p\, \underline{V} \cdot d\underline{S}$

why the γ ? → outward flow of thermal energy $\left(d\underline{S} \cdot \underline{V} \frac{p}{\gamma-1}\right)$ thru S

$p\,dV$ work of blob on exterior

③

as $\underline{E} = -\frac{\underline{V}\times\underline{B}}{c}$ . . . →

so ③ $= d\underline{S} \cdot \frac{\underline{E}\times\underline{B}}{4\pi c}$ → $\left\{ \begin{array}{l} \text{loss of} \\ \text{energy by} \\ \text{Poynting flux} \end{array} \right.$

④ loss of gravitational potential energy due outflow from blob ...

It's all clear !!

this brings us to ....

→ **Virial Theorems in MHD**

- what is a virial theorem
- why yet another theorem ?

→ Virial Theorems are:

- space/time averaged energy theorems
- "lumped parameter" relations for energies in complex, multi-element interacting systems
- useful for 'back-of-envelope' estimates, etc.
- logically extend the moment program:

$$\underbrace{f(\underline{x}, \underline{v}, t)}_{\text{phase space fluid}} \xrightarrow[\underline{v}\text{ moments}]{} \underbrace{n(\underline{x}, t), \underline{v}, T}_{\text{position space fluid}} \xrightarrow[\text{virial integrals}]{} \underbrace{E_v, \ E_B, \ \text{etc.}}_{\#}$$

space integrals

THE Question for Virial Theorem

Before proceeding:

$\boxed{\text{?! Can an isolated blob of MHD plasma confine itself without self gravity ?!}}$

Easily answered by $\boxed{\text{Virial Theorem}}$...

Recall, for system of particles, Virial theorem derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i\cdot\underline{x}_i\right) = \sum_i \underline{p}_i\cdot\dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i\cdot\underline{x}_i$$

$$= \underbrace{2T}_{\text{kinetic energy}} + \underbrace{\sum_i\left(-\frac{\partial U}{\partial \underline{x}_i}\right)\cdot\underline{x}_i}_{\text{via Newton's Law}}$$

$$\langle\ \rangle = \int_0^T \frac{dt}{T}, \quad T\to\infty$$

Now, if $\boxed{\sum_i \underline{p}_i\cdot\underline{x}_i \ \text{bounded}}$ (blob),

$$\left\langle \frac{d}{dt}\sum_i \underline{p}_i\cdot\underline{x}_i\right\rangle = \frac{1}{T}\int_0^T dt\,\frac{d}{dt}\left(\sum_i \underline{p}_i\cdot\underline{x}_i\right) \ \underset{T\to\infty}{\longrightarrow} 0$$

so ...

$\to$ (first) Virial of system

$$2\langle T\rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i}\cdot \underline{x}_i \right\rangle$$

Further, if $U = U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n)$

where $U(\alpha\underline{x}_1, \alpha\underline{x}_2, \cdots \alpha\underline{x}_n) = \alpha^k U(\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n)$

homogeneous function

(scaling $\Leftrightarrow$ structure of power-law potentials $\to$ i.e. h.o. $\to k=2$, Coulomb $\to k=-1$)

Scaling!

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}$$

but of course:

$$T + U = \langle T\rangle + \langle U\rangle = E$$

then $\left(\frac{k}{2}+1\right)\langle U\rangle = E$

$$\boxed{\langle U\rangle = \frac{2}{k+2}E}, \quad \boxed{\langle T\rangle = \frac{kE}{k+2}}$$

check: $k=2$, $\langle U\rangle = (1/2)E$, $\langle T\rangle = (1/2)E$ ✓

$k=-1$ (gravity), $\langle T\rangle = -E$ ✓ $(\Rightarrow E<0)$ $\Rightarrow$ bounded motion only if total energy negative (i.e. bound state)

Aside: Simplest realization of negative specific heat 'paradox', i.e.

 → consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  → E decreases ↔ R decreases

R ↓

$\therefore$ $(-E)$ increases ⇒ $\langle T \rangle$ increases
↳ kinetic energy

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling ⇒ [temperature increase]!!

$$\Rightarrow \underset{\text{specific heat}}{c} < 0 \text{ !!}$$

[In the days before the discovery of nuclear fusion, this was thought to be what 'heated' stars. Kelvin, in particular, was a ~~prop~~ proponent.

Now, proceeding to full Virial theorem ...

→ Consider equation of motion:

$$\begin{cases} \underbrace{\frac{\partial}{\partial t}(\rho v_i)}_{\text{momentum}} = -\frac{\partial}{\partial x_j} \underbrace{T_{ij}}_{\underline{\text{full}}\ \text{stress tensor}} \\ T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij} \end{cases}$$

Now, recalling relation of **Virial to $\frac{d}{dt}(\underline{p}\cdot\underline{x})$**
⇒ consider:

Moment inertia / distrib.

$$\boxed{I_{ij} = \int d^3x\, \rho\, x_i x_j}$$

(~ $\underline{\text{moment of inertia}}$ distrib)

↳ Virial theorem is for tensor ---

and

$$\frac{d}{dt} I_{ij} = \int d^3x \frac{\partial \rho}{\partial t} x_i x_j$$

$$= -\int d^3x \frac{\partial}{\partial x_t}(\rho v_t)\, x_i x_j$$

integrating by parts assuming ρ compact (i.e. 'blob' of interest)

$$= \int d^3x \left[\rho x_i v_j + \rho x_j v_i\right]$$

so

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i \left(\frac{\partial}{\partial t}\rho v_j\right) + x_j \frac{\partial}{\partial t}(\rho v_i)\right]$$

but $\frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming $\left\{ \boxed{\text{compact blob, no external linkage}} \right.$

$\Rightarrow$

$$\frac{d^2 I_{i,j}}{dt^2} = +\int d^3x \left[\delta_{i,t} T_{j,t} + \delta_{j,t} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$ unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as $\underline{T_{ij}}$ manifestly symmetric $\Rightarrow$

$$\boxed{\begin{aligned} \frac{1}{2}\frac{d^2 I_{ij}}{dt^2} &= +\int d^3x\, T_{ij} \\ T_{ij} &= \rho V_i V_j + \left(p + \frac{B^3}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\delta_{ij} \end{aligned}}$$

— tensor virial theorem.

note unlike simple pt particle example, time dependence remains.

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I = I_{ii} = \mathrm{tr}\, I_{ij}$$

repeated indexes summed

$\mathrm{tr}\,(\text{V.T.}) \Rightarrow$

$$\mathrm{tr}\ \frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\, \frac{\rho x^2}{2}\right)$$

$$= \mathrm{tr} \int d^3x \left[\rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\, \delta_{ij}\right]$$

$$= \int d^3x \left[\rho v^2 + 3\left(p + \frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi\right]$$

$\therefore \quad I \equiv \int d^3x\, \rho x^2/2 \quad \Rightarrow$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x \left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi\right]}$$

$\rightarrow$ **Scalar Virial Theorem**

Now, first neglect self-gravitation $\Rightarrow$

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho v^2 + 3p + B^2/8\pi\right]$$

Now $\rightarrow$ can an isolated blob of MHD fluid confine itself ??

If 'self-confined' $\Rightarrow$ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent $\Rightarrow \dot{I}, \ddot{I} = 0$

stable pulsation $\Rightarrow \ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho v^2 + 3p + B^2/8\pi\right]$

so even if $v^2 = 0$ (no fluid motion in blob) $\Rightarrow$

$p > 0$, $B^2/8\pi > 0$ $\Rightarrow \ddot{I} > 0$!

$\therefore$ $\boxed{\text{No} \rightarrow \text{isolated blob can't confine itself.}}$

More generally, noting that

$$E_V = \int d^3x\, \rho V^2/2$$

$$E_P = \int d^3x \frac{P}{\gamma - 1} = \frac{3}{2}\int d^3x\, P \qquad \text{(gas)}$$

$$E_B = \int d^3x \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2 I}{dt^2} = 2E_V + 2E_P + E_B}$$

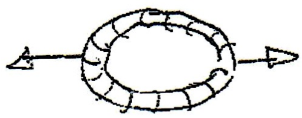
simple relation in terms energies.

Aside: $\Rightarrow$ So, isolated blob can't confine itself

$\Rightarrow$ how is
- tokamak $\rightarrow$ $\underline{B_T}$ for stability; not transport
- or - better macro-confinement
- RFP $\rightarrow$ weak external B_T guide (negligible)

confined ?? Confinement by wall is unacceptable ...

Answer: → toroidal plasma tends to expand toroidally



→ held in place by $\begin{cases} \text{conducting shell (often undesirable) or} \\ \text{"vertical field"} \end{cases}$

i.e.  → additional external P_{Mag} to oppose toroidal expansion – vertical field

→ image currents in close-in conducting shell can do likewise.

JET anecdote
re: vertical field failure ...

Now, retaining self-gravitation:

$$T_{ij}\Big|_{gravity} = \rho\phi\,\delta_{ij} = 2\underbrace{\left(\frac{\rho\phi}{2}\right)}_{\mathcal{E}_{gravity}}\delta_{ij}$$

to calculate: $\nabla^2\phi = 4\pi G\rho$

$$\Rightarrow \quad \phi = -G\int d^3x\, \frac{\rho(x')}{|x-x'|}$$

so

$$T_{ij}\Big|_{\text{gravity}} = T\Big|_{\text{gravity}}\delta_{ij}$$

$\Rightarrow$

$$T = -\frac{G}{2}\int d^3x \int d^3x' \frac{\rho(\underline{x})\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= + E_{\text{gravitation}} \equiv \quad E_g < 0$$

so scalar Virial theorem becomes, with gravity $\Rightarrow$

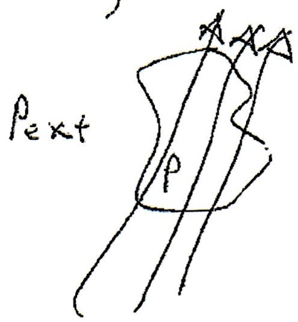
$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_V + 2E_P - |E_g| \quad + E_B}$$

so with gravity, can have self-confining blob
(no surprise ...)

> This brings us to another application of Virial theorems, namely proto-stellar cloud collapse ....

What?

– now, consider a plasma cloud/blob



- mass M, radius R
- threaded by $\underline{B}$
- pressure P, external pressure P_0
- no bulk motion
- frozen flux

now, easy to show for $\ddot{I} = 0$, $\underline{V} = 0$, must have:

surface terms

$$\left[2E_p - |E_g| + E_B \right] = \int dA\, \underbrace{P_{ext}}_{\text{external pressure}}\, \hat{x}\cdot\hat{n} - \int dA\, \underline{x}\cdot \underbrace{\underline{\underline{T}}_B}_{\text{magnetic stress thru surface (threading fields)}} \cdot \hat{n}$$

Now, can estimate:

$M = \int \rho \, dV \rightarrow$ total mass $\rightarrow \rho R^3$

$E_p \cong C_s^2 M$

$|E_g| \cong \underbrace{\alpha}_{\text{form factor}} \frac{GM^2}{R}$

– For frozen flux, $\Phi \sim \pi R^2 B$

$B \sim \frac{\Phi}{R^2}$

$R^3 B^2 \sim \Phi^2/R$

so $\quad E_B + \oint dA\, \underline{x} \cdot \underline{\underline{T}}_B \cdot \hat{n} \sim \beta \Phi^2/R$

$\Rightarrow$ have: (eliminating extraneous factors):

$$\boxed{R^2 P_{ext} \sim \left(\frac{\beta \Phi^2}{R} - \alpha \frac{G M^2}{R} + \frac{3}{2} C_s^2 M \right)}$$

$\rightarrow$ scalar virial theorem for cloud....

Now: $\quad P_{ext} \sim \left(\frac{\beta \Phi^2}{R^3} - \alpha \frac{G M^2}{R^3} + \frac{3}{2} \frac{C_s^2 M}{R^2} \right)$

$\rightarrow$ if Φ, $G \rightarrow 0$ $\rightarrow$ need $P_{int} = P_{ext}$ for confinement.....

$\rightarrow$ if $\Phi = 0$

$$P_{ext} = -\frac{\alpha G M^2}{R^3} + \frac{3}{2} \frac{C_s^2 M}{R^2}$$

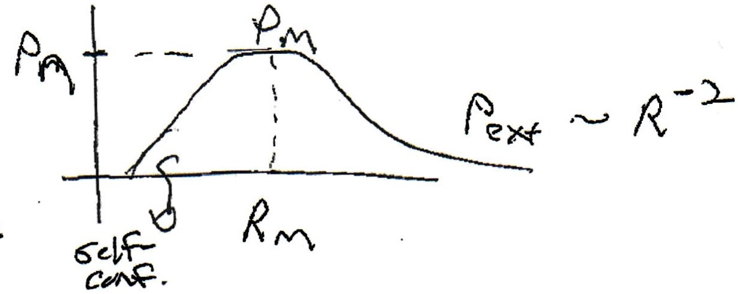
$dP/dR = 0 \quad \Rightarrow \quad 3 \alpha \frac{G M^2}{R^4} = 3 \frac{C_s^2 M}{R^3}$

Radius at which P_{ext} max

$$\boxed{R_{max} = G M \alpha / C_s^2}$$

[Note: $\Rightarrow R_m^2 = (G\rho/c_s^2)^{-1/2} \Rightarrow L^2_{Jeans}$]

so P_{ext} needed



- $P > P_{max}$ → no equilibrium
- $R < R_{max}$ → P_{ext} must decrease to maintain equilibrium ⇒ instability to gravitational collapse!

- $\Phi \neq 0$ (magnetic field on ...) → note immediately that magnetic support scales similarly to gravitational attraction, with opposite sign

⇒

$$P_{ext} \sim \left[\left(\beta \Phi^2 - \alpha G M^2\right)/R^3 + \frac{3}{2} \frac{C_s^2 M}{R^2} \right]$$

so key point is: $\boxed{\left(\beta \Phi^2 - \alpha G M^2\right) \lessgtr 0 \,?!}$

$$\Rightarrow M \gtrless M_\Phi = \sqrt{\beta/\alpha}\, \Phi / G^{1/2}$$

- $\boxed{M < M_\Phi}$ → magnetically subcritical mass for gravitational collapse

$M > M_\Phi$ → magnetically super-critical mass for collapse!

i.e. $M < M_\Phi$ → repulsive effects $\left\{\begin{array}{l}\text{field}\\ \text{thermal}\end{array}\right\}$ pressure always win
$(M_\Phi^2 - M^2 > 0)$ → no ammount of external compression can induce indefinite contraction, IF flux remains frozen in

$M > M_\Phi$ → sufficient external pressure/compression can induce gravitational collapse, even if flux frozen in.
$(M_\Phi^2 - M^2 < 0)$

[Note: If kinetic energy contribution, NL Alfven waves can support cloud.]

For perspective, recall:

- (famous) Chandrasekhar Mass
  - $M > M_{Chandra}$ → collapse
  - $M < M_{Chandra}$ → no collapse.

$M_{Chandra}$ derived for degenerate Fermi gas equation of state → $\gamma = 4/3$, instead of $\gamma = 5/3$.

- of flux-freezing $\Rightarrow$ $\Phi \sim B R^2$
$\rho R^3 \sim M$

$\Rightarrow B \sim R^{-2} \Rightarrow B \sim \rho^{2/3}$

$\therefore \boxed{B^2 \sim P_{Mag} \sim \rho^{4/3}}$

$\Rightarrow$ <u>if</u> flux frozen, field obeys equation of state like Fermi gas

(i.e. Flux freezing is akin to exclusion, albeit on field-lines-per-fluid-element)

$\therefore$ an analogue to Chandrasekhar mass seems quite plausible ....

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

$\therefore$ $n_{\text{fermion}} \sim N/R^3$

$\therefore$ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$p \sim \hbar/\Delta x \sim \hbar n^{1/3}$ (Heisenberg Uncertainty)

p → Fermion Momentum

⇒ Fermion energy (per Fermion) : $E_F = pc$
$\sim \hbar c \dfrac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

Gravitational Energy (per Fermion) : $E_{Grav} \sim -\dfrac{GMm_b}{R}$

m_b → Baryon mass

$M \sim N\, m_B$

Pressure → electron
Mass → Baryon

$\therefore$ $E = E_F + E_G$

$= \dfrac{\hbar c N^{1/3}}{R} - \dfrac{GNm_B^2}{R}$

note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,
$\therefore E_F \sim 1/R^2 \to$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse.

$\therefore$ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore \quad M_{Chandra} = N_{max} m_B \sim 1.5 M_\odot$$