Lect. II — Freezing-in Law

→ Recall

MHD: EOM, O.L. + 3 Maxwell

$\boxed{\text{Cont. Eqn State}}$

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \frac{\underline{J}\times\underline{B}}{c} + \underline{f}$$

$$= -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi} + \underline{f}$$

$$\boxed{\partial_t \underline{B} - \eta\nabla^2\underline{B} = \nabla\times\underline{v}\times\underline{B}}$$

Induction.

Faraday + Ohms

→ 2 strongly coupled, interpenetrating fluids

$\frac{\partial \underline{v}}{\partial t} + \ldots$ $\qquad$ $\underline{v}, \underline{B}$

$\frac{\partial \underline{B}}{\partial t} + \ldots$

→ even clearer for incompressible.

⇒ the induction equation for $\underline{B}$ evolution ...

$$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}}$$

Induction eqn.

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) — $\underline{V}(\underline{x}, t)$, $\underline{B}(\underline{x}, t)$ evolving simultaneously

↓

- useful and instructive to re-write induction equation

$$\nabla \times \underline{V} \times \underline{B} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

so

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}.$$

This brings us to ......

→ What Does "MHD", as a system, really mean ....?

this is answered most clearly for the case of
incompressible MHD ---

$\nabla \cdot \underline{V} = 0$ → defines equation of state
→ sets P_{total} field

$\boxed{(\omega/k \ll C_s, V_{MS})}$ $|\tilde{V}| < C_s, V_{Alfv}$

sound, magnetosonic

$$\nabla \cdot \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\frac{\nabla}{\rho}\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi\rho} \right\}$$

$$\frac{d\rho}{dt} = -\rho\,\nabla\cdot\underline{V} = 0$$

so $\rho \to$ constant ρ_0 (can relax to slow variation)

$$\nabla^2\left[\left(P + \frac{B^2}{8\pi}\right)\Big/\rho_0\right] = \nabla\cdot\left(\frac{\underline{B}\cdot\nabla\underline{B}}{4\pi\rho_0} - \underline{V}\cdot\nabla\underline{V}\right)$$

total pressure

a la' Poisson's equation:

$$\frac{P + B^2}{8\pi} = -\int \frac{d^3x'}{4\pi|x - x'|}\left\{\nabla'\cdot\left(\frac{\underline{B}\cdot\nabla'\underline{B}}{4\pi\rho_0} - \underline{V}\cdot\nabla'\underline{V}\right)\right\}$$

solves for: P_{tot} field → eliminates eqn. state

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\frac{\underline{\nabla} p^*}{\rho_0} + \frac{\underline{B} \cdot \underline{\nabla}\, \underline{B}}{4\pi\rho_0}$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{v}$$

Basic MHD $\boxed{\nabla\cdot\underline{V}=0}$ $p^* = p_{tot}$

$$\boxed{\begin{aligned} &\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\frac{\nabla}{\rho_0}\left(p^*\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi\rho_0} \\ &\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\nabla\underline{B} - \eta\nabla^2\underline{B} = \underline{B}\cdot\nabla\underline{V} \end{aligned}}$$

with $\nabla\cdot\underline{V}=0$, constitute equations of incompressible MHD.

→ Rather clearly, this system is one of two dynamically coupled, evolving vector fields $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$.

→ $\boxed{\text{Compressible MHD}}$ is really a problem in 3 fields, two of which are vectors

i.e.

- $\underline{V}(\underline{x},t)$ → fluid velocity
- $\underline{B}(\underline{x},t)$ → magnetic field
- $S(\underline{x},t)$ → entropy ⇒ energy density

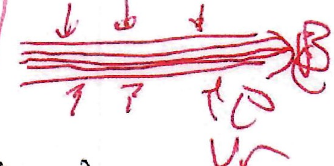
i.e. scalar equation of state provides 3rd field.

→ Key Question: $\boxed{\text{How closely coupled are } \underline{V}, \underline{B} \text{ ?}}$

⇒ the key physics element in MHD ....

⇒ $\boxed{\text{Frozen-in Law, Flux Freezing}}$

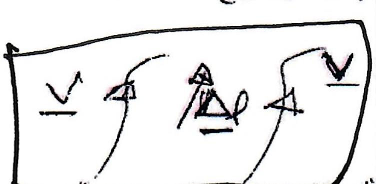
2 versions
→ local
→ integrated



B scale

① Frozen-in Law

- consider a (for the moment, passive) vector field:
  - frozen into flow $\underline{V}(\underline{x}, t)$
  - consisting of oriented, flexible strands



$\underline{\Delta \ell}$ — oriented length

$\underline{\ell_0} + \underline{\Delta \ell}$

$\underline{\ell_0}$

i.e. massless rubber strands in flow

How does $\underline{\Delta \ell}$ evolve?

in dt,
$$d(\underline{\Delta \ell}) = \left(\underline{V}(\underline{\ell_0} + \underline{\Delta \ell}) - \underline{V}(\underline{\ell_0})\right) dt$$
$$= \underline{\Delta \ell} \cdot \nabla \underline{V} \quad dt$$

$$\therefore \quad \frac{d(\underline{\Delta \ell})}{dt} = \underline{\Delta \ell} \cdot \nabla \underline{V}$$

$$\frac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell}\cdot\underline{\nabla V}$$

i.e. $$\boxed{\frac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell}\cdot\underline{\underline{S}}}$$

$$\left\{\begin{array}{l} \dfrac{d}{dt}(\Delta \ell)_i = \Delta \ell_j \cdot \underline{\underline{S_{ij}}} \\[2ex] S_{ij} = \dfrac{1}{2}\left(\dfrac{\partial V_i}{\partial x_j} + \dfrac{\partial V_j}{\partial x_i}\right) \end{array}\right.$$

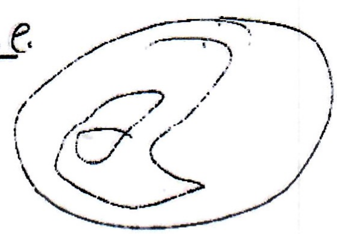
→ [rate of strain tensor

→ strain rate tensor

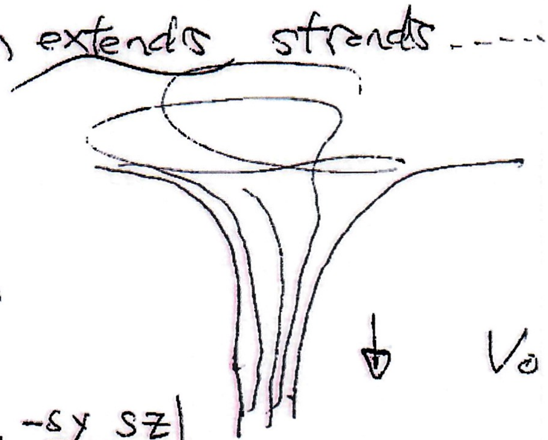
says that → $\underline{\Delta \ell}$ strands orient along strain

→ strain extends strands ....

i.e.



→ siphon flow



$$\underline{V} = V_0\left(-\frac{Sx}{2}, -\frac{Sy}{2}, Sz\right)$$

plausible to say that $\underline{\Delta \ell}$ "frozen into" the flow.

Now, if $\eta \to 0$, ...in MHD

$$\boxed{\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V}}$$

$$-\underline{\nabla}\cdot\underline{V} = +\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{\partial}{\partial t}\underline{B} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \frac{d\underline{B}}{dt} = \underline{B}\cdot\underline{\nabla}\,\underline{V} + \frac{\underline{B}}{\rho}\frac{d\rho}{dt}$$

$$\frac{1}{\rho}\frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2}\frac{d\rho}{dt} = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}V$$

$$\therefore \quad \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}\,\underline{V}}$$

→ $\underline{B}/\rho$ obeys same equation as $\underline{\Delta l}$!

→ $\boxed{\underline{B}/\rho \text{ is frozen into flow field } \underline{V}(x,t)}$

* Note: → $\underline{B}/\rho$ is **not** passive – due $\underline{J}\times\underline{B}$ force

→ $\underline{B}$ determines flow, while frozen into it!

→ (essence of coupling problem)

→ Strength of Freezing-in ?

$$\frac{\partial \underline{B}}{\partial t} = \underbrace{\nabla \times \underline{v} \times \underline{B}}_{①} + \underbrace{\eta \nabla^2 \underline{B}}_{②}$$

$$①/② \cong VB/L \Big/ \frac{\eta B}{L^2} \sim \underbrace{\frac{VL}{\eta}}$$

Magnetic Reynolds #

- $R_M \gg 1$ → field frozen in ~

- departure $\ell \sim L_0 / R_M$. ⟨ small scale
  better? - reconnection

- Model need not be resistive MHD

- For $\underline{\nabla} \cdot \underline{V} = 0$, $\underline{B}$ frozen in

⇒ if $\eta \neq 0$, freezing in is broken ----

i.e.
$$\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) - \frac{\eta}{\rho} \nabla^2 \underline{B} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V}$$

form of frozen evolution broken

[Vorticity connection

⇒ Observe: → this motivates attention to resistivity in MHD above other dissipations ν, χ, etc..

→ η ⇒ $\underline{B}$ diffusion $\sim \eta \nabla^2$

∴ decoupling of $\underline{V}$, $\underline{B}$ occurring on small scales

⇒ [motivates 'magnetic reconnection' as study of singularity dynamics in MHD.] ⇒

→ A Word to the Wise: In modelling, describing complex dynamics in MHD (i.e. MHD turbulence, dynamos, etc.) always think carefully about frozen-in law ...

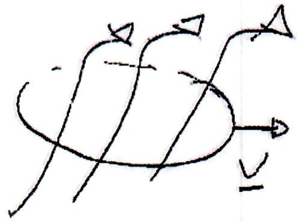
What is frozen in

[illegible] ?

Alfven's Thm

→ Closely Related: Flux Freezing

- consider flux thru surface in flow



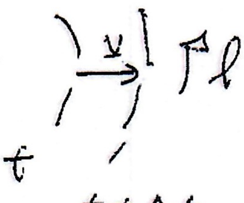
→ i.e. imaginary loop drawn in flow field ...

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

$$\underline{\Phi} = \int \underline{B} \cdot d\underline{s}$$

① change in $\underline{B}$ ② change in $d\underline{s}$

$$\frac{d\underline{\Phi}}{dt} = \underbrace{\int d\underline{s} \cdot \frac{\partial \underline{B}}{\partial t}}_{\text{change in } \underline{B}} + \underbrace{\int \frac{d\underline{s}}{dt} \cdot \underline{B}}_{\text{motion of loop...}}$$

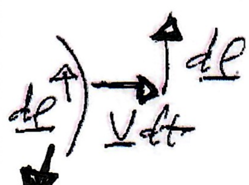


$$① = \int d\underline{s} \cdot \underline{\nabla} \times (\underline{V} \times \underline{B}),$$

$$= \oint d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$

$$ds = \underline{V} \Delta t \times d\underline{\ell}$$

For ②



so $\boxed{d\underline{s} = \underline{V}dt \times d\underline{\ell}}$

↳ {change in $\underline{s}$ in dt.

$dt\,(d\underline{\Phi}_{②}/dt)$

$$②dt = \int (\underline{V}dt \times d\underline{\ell}) \cdot \underline{B} = d\underline{\Phi}$$

$$\left.\frac{d\underline{\Phi}}{dt}\right)_{②} = \int (\underline{V} \times d\underline{\ell}) \cdot \underline{B} = -\int d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$

so $\boxed{\dfrac{d\Phi}{dt} = ① + ② = 0}$ ✓

so $\Rightarrow$ magnetic flux invariant $\Leftrightarrow$ cancellation

$\Rightarrow$ in absence of resistivity, flux thru surface in flow is invariant, or frozen in

$\Rightarrow$ no surprise: $\underline{B}$ frozen in $\Rightarrow$ Φ frozen in

$\Rightarrow$ analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v} \cdot d\underline{l} = \int d\underline{a} \cdot \underline{\omega} \qquad \omega = \nabla \times \underline{v}$$

In inviscid hydro, $(\nu \to 0)$ circulation Γ_c is conserved. $P = P(\rho)$

[Exercise: Prove this!
Note relation between $\underline{\omega}$ equation and $\underline{B}$ eqn. Assume $\rho = \text{const}$, $\underline{g} = 0$.]

[Extra Credit: ① Discuss the extension to the case where $\rho \neq \text{const}$.
② what is 'frozen in' for Vlasov plasma?]

→ PV is macroscopic (unlike f)
yet conserved

and → Point of H-M is that
Drift Wave Turbulence is 'like'
Geophysical Fluid Turbulence

and
→ GFD makes heavy use of PV.

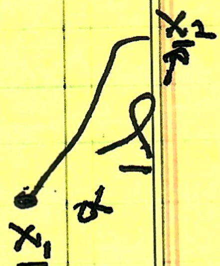
indeed: "GFD = Fluid Dynamics of PV"
Likewise plasma ...

Key Point: $Ro \ll 1$
Rossby #

$Ro \sim V_\perp / L_\perp \Omega$ (Fluid)

$Ro \sim V_\perp / L_\perp \Omega_i$ (Plasma)

@ 2D flow
⇒ vorticity along rotation axis is key
→ what?



$$\frac{d\underline{x}}{dt} = \underline{V}(\underline{x})$$

"points frozen into flow"

Can consider $\underline{\ell}$, a line segment $\underline{\ell}$,
(i.e. flexible monofilament inserted into
the flow ...)

$$\frac{d}{dt}\underline{\ell} = \underline{V}(\underline{x}_2) - V(\underline{x}_1)$$

$(\underline{x}_2 - \underline{x}_1)/2 \equiv \underline{\ell}$

so

$$\boxed{\frac{d\underline{\ell}}{dt} = \underline{\ell} \cdot \underline{\nabla V}}$$

$\underline{\ell}$ "frozen-in" to the flow.

N.B.:

→ $\frac{d\ell}{dt} \neq 0$ ⇒ allows for stretching

→ obvious similarity to $\underline{B}/\rho$ in ideal MHD (local form Alfven's Theorem),

$$\boxed{\frac{d}{dt}\underline{B}/\rho = \frac{\underline{B}}{\rho} \cdot \underline{\nabla V}}$$

$\frac{\underline{B}}{\rho}$ "frozen in"

eqns. ℓ, $\underline{B}/\rho$ same form

→ For flow (i.e. dons, for plasma)

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{V} = -\frac{\underline{\nabla} P}{\rho} - \underbrace{2\underline{\Omega} \times \underline{V}}_{\text{Coriolis / Lorentz}}$$

Then seek vorticity (Plasma/GFD all about vorticity)

Why? → vorticity along $\langle \underline{B} \rangle$ describes dynamics

→ ≈ 2D.

$\underline{\omega} = \underline{\nabla} \times \underline{V}$

then → $\underline{V} \cdot \underline{\nabla} \, \underline{V} = -\underbrace{\underline{\nabla}\left(\frac{V^2}{2}\right)}_{\text{dynamic pressure}} - \underbrace{\underline{V} \times \underline{\omega}}_{\text{Magnus Force}}$

→ $P = P(\rho)$
otherwise, enter Ertel's Thm.
and non-conservation of circulation.
i.e. $P(\rho, T) \rightarrow \nabla\rho \times \nabla T$ drive
see Mueller. (good Paper Topic)

then,

$$\partial_t (\underline{\omega} + 2\underline{\Omega}) = \underline{\nabla} \times \left[\underline{V} \times (\underline{\omega} + 2\underline{\Omega}) \right]$$

and, since $\underline{\Omega}$ $\begin{cases} \text{static} \\ \text{uniform} \end{cases}$ $\underline{\Omega} = \Omega \hat{z}$

$$+ \quad \frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{V}) = 0$$

⇒ Freezing in law for vorticity:

$$\frac{d}{dt}\frac{(\underline{\omega}+2\underline{\Omega})}{\rho} = \frac{(\underline{\omega}+2\underline{\Omega})}{\rho}\cdot\underline{\nabla}\,\underline{V}$$

obviously: ⇕

$$\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \left(\frac{\underline{B}}{\rho}\right)\cdot\underline{\nabla}\,\underline{V}$$

, Ideal MHD

$$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = 0$$

so $\dfrac{\underline{\omega}+2\underline{\Omega}}{\rho}$ frozen-in!

N.B. For $\begin{cases}|\Omega| \gg \text{other rates in problem}\\ \rho \simeq \text{const}\end{cases}$

⇒ $\underline{\Omega}\cdot\underline{\nabla}\,\underline{V} \cong 0$ ⇒ Taylor-Proudman Theorem

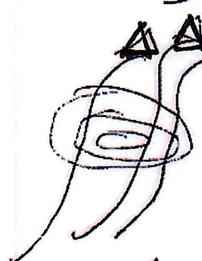
i.e. flow uniform along direction of rotation axis

⇒ 2D-ized!

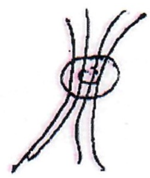
N.B.: obviously, frozen-in ≠ passive!

→ What Does "Freezing" Mean?

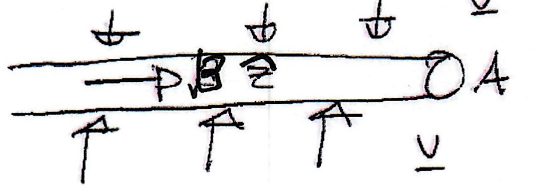
→ can relate field evolution in a flow to density evolution, since $\underline{B}/\rho$ is "frozen in".

Application: 

Star formation

$\underline{B}$-field →  (?)

Simple Cases → How does $\underline{B}$ change in a flow?

(i) 

radial compression

cylinder of cross-sectional area A squeezed in radial flow. $\underline{B} \equiv B\,\hat{z}$

2 ways:

$$\frac{d}{dt}\left(B\hat{z}/\rho\right) = \frac{B\hat{z}\cdot\nabla\,\underline{V}}{\rho}$$

$\underline{V} = v\,\hat{r}$

$\Rightarrow \underline{V} \perp \underline{B}$

$$= 0$$

so $\underline{B}/\rho \doteq$ const

Now! $\rho A L$ = const so $B \sim A^{-1}$
$\rho \sim A^{-1}$
(L const)

or

Flux Frozen: $BA = \Phi$ = const.
$\rho A L$ = const $= M$
L const.

$BA \sim \Phi_0$, $B \sim A^{-1}$
$\rho A \sim M_0$, $\rho \sim A^{-1}$
so $B \sim \rho^{(1)} \Rightarrow B/\rho \sim$ const!

ii.) $V = V(z)\hat{z}$ – compressible!
$\longrightarrow B\hat{z}$ i.e. stretch, 1D

here $\frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V} \neq 0$, but easier to work with $\underline{B}$ than $\underline{B}/\rho$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{v} - \underline{B}\, \underline{\nabla} \cdot \underline{v}$$

$$= B \frac{\partial}{\partial z} V(z) - B \frac{\partial V(z)}{\partial z}$$

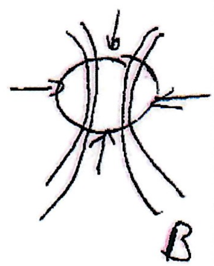
$$= 0 \;!$$

For ρ, $\quad \frac{d\rho}{dt} = -\rho\, \underline{\nabla} \cdot \underline{V} = -\rho \frac{\partial V_z}{\partial z}$

here $\underline{B}$ invariant, ρ changes $\qquad \frac{d}{dt}\frac{\underline{B}}{\rho} = \frac{\underline{B}}{\rho} \cdot \nabla \underline{V}$

i.e. $\quad B \sim \rho^{(0)}$ $\qquad$ Freezing in $\not\Rightarrow$ const

iii.)



collapsing sphere: $\underline{V} = V \hat{r}$

(i.e. $\Phi = 0$, total sphere)

consider hemispherical surface (i.e. mushroom cap)

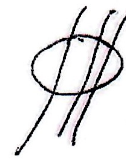
$$\boxed{\begin{aligned} \Phi &\sim B R^2 \\ M &\sim \rho R^3 \end{aligned}} \begin{aligned} &\sim \text{const} \\ &\sim \text{const.} \end{aligned}$$

$$\Rightarrow \begin{aligned} B &\sim r^{-2} \\ \rho &\sim r^{-3} \end{aligned} \qquad \Rightarrow \boxed{B/\rho^{2/3} \sim \text{const.}}$$

why the scaling ? $\Leftrightarrow$ why of interest ?

→ $\left\{\begin{array}{l}\text{"implosion"} \\ \text{gravitational collapse} \\ \text{equation of state}\end{array}\right\}$ problems sensitive to equation of state of material collapsing

$\underline{\underline{\text{If}}}$: $p \to P_{tot} = p + \dfrac{B^2}{8\pi}$



collapse threaded by magnetic field

$$P = P_0 \left(\rho/\rho_0\right)^{\gamma}$$

then natural to ask: Can one write $B^2 = B^2(\rho)$ and thus extend equation of state to encompass magnetic pressure contribution?

⇒ proceed via flux-freezing!

$$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$$

$\underline{so}$ P_{B^2} has "γ_{eff}" $= 4/3$. This resembles equation of state for degenerate gas (see Handouts I).

⇒ More on this in discussion of flux freezing and $\underline{\text{Virial}}$ $\underline{\text{theorems}}$ ....

→ $\underline{\text{Pragmatic Question}}$: Is flux "frozen" during star formation? ↔ Does resistivity matter?

$\eta \sim \frac{4 \times 10^{6}}{T_{eV}^{3/2}}$ cm²/sec. (Spitzer)

start → collapse → protostar

$n \sim 1$ atom/cm³ ⟶ $\rho \sim 1$ g/cm³
$n \sim 10^{24}$ atm/cm³
(related N_A)

but

$B/\rho^{2/3} \sim$ const

$\Rightarrow \quad B/B_0 \sim (10^{24})^{2/3} \sim 10^{16}$! huge amplification

so $B_0 \sim 10^{-6}$ G, characteristic of ISM

$\Rightarrow \quad B \sim 10^{10}$ G in protostar

$\therefore \quad P_{B^2} \sim 10^{19}$ erg/cm³ $\quad (P_{B^2} \sim B^2/8\pi)$

but P_{Th} for normal star $\sim 10^{14}$ erg/cm³

$P_{B^2} \gg P_{Th}$?? $\Rightarrow$ clearly flux-freezing is bad assumption.

→ In terms of time scales:

$$\frac{\partial B}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) + \eta \nabla^2 B$$

① ② ③

$$\frac{1}{T_{collapse}} \sim \frac{1}{T_{dynamic}} + \frac{\eta}{L^2}$$

$\approx 1/T_{diffn.}$

3 scales, 2 balance

i.e. ① & ② ③ negligible

① & ③ ② negligible.

if $T_{collapse} \ll T_{diffn} \rightarrow$ {flux frozen, OK

$T_{collapse} \gg T_{diffn} \rightarrow$ {must consider diffusion, freezing invalid

N.B.: In star formation, $T_{coll.} \ll T_{diffn}$

but ISM has large neutral component

Plasma-neutral drag sets dissipation

→ Ambipolar diffusion.